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Laminar Bingham fluid flow between vertical parallel plates

Yildiz Bayazitoglu [a,*], Paul R. Paslay [b], Paul Cernocky [c]

[a] *Mechanical Engineering and Materials Science Department, Rice University, Houston, TX, USA*
[b] *Techaid Corporation, Arlington, TX, USA*
[c] *Shell International Exploration & Production Inc., Houston, TX, USA*

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Abstract

Packer fluids for deep-water oil and gas wells are being developed currently to minimize the rate of heat transfer from the flowing production fluid to the outer casing annuli. In this work we study a gel that has a yield point capable of preventing or drastically reducing natural convective fluid flow and therefore the heat transfer that otherwise would occur from the production tubing to the production casing. The gel is modeled as a Bingham material. The tubing-to-casing annulus is geometrically modeled as vertical and large parallel plates. This modeling is appropriate as the radial extent of the annulus containing the fluid is usually small compared to the mean radius of the annulus. The flow is assumed to be laminar, and in order to provide a reference case the solution to the linear viscous flow is first presented. The natural convection problem of the Bingham fluid is described in five distinct regions within the gap between parallel plates, progressing from hotter to the cooler plate. The velocity and shear stress distributions with some examples from the oil industry are given. Equivalent dimensionless numbers are developed for the Bingham fluid in order to be able to use the available linear viscous correlation equations. The correlation results characterize the heat transfer performance of the gel.

Keywords: Bingham fluid; Natural convection; Vertical parallel plates; Laminar flow; Completion fluid

1. Introduction

The results presented here are the first part of a more comprehensive investigation. The motivation for this investigation is twofold: (i) prevention of heat transfer which would cause heating and the buildup of large pressures in fluids trapped in outer annuli between the layered casing strings of oil and gas wells; and (ii) prevention of the occurrence of gas hydrates, paraffin deposits and asphaltene formation in vertical production lines. The buildup of trapped pressures in the fluids between casing annuli can lead to collapse of multiple casings and loss of the entire well. This problem occurs because of too much heat being transferred from the production tubing to the surrounding casings. Gas hydrate, paraffin and asphaltene precipitation are the result of cooling the production fluid below critical temperatures, and these can lead to plugging of the well production conduit [1]. This problem also occurs because of too much heat being lost to the surroundings. Consequently, methods of minimizing heat transfer between hot production fluids and outer, initially cold layers of the well are of considerable practical interest, and various mechanical and fluid approaches are being studied and used to achieve this purpose.

The oil industry is interested in using gels as packer fluids in the annulus between the production tubing and the first (production) casing in order to significantly reduce the heat transfer rate to the outer casing annuli compared to having seawater in the annulus behind the tubing. These special fluids can have rather large gel strengths which can be detrimental if a casing has to be removed as the fluid must be pumped out of the well. Such fluids have been used with mixed success over a period of thirty years both in producing wells and in steam injection wells. The oil and gas industry sometimes tests these non-convecting packer in large scale heating cells or in direct, instrumented well performance to determine whether or not the fluid provides the expectation of no-convection; but little work has been done to fundamentally model and characterize the convective heat

* Corresponding author.
E-mail address: bayaz@rice.edu (Y. Bayazitoglu).

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Nomenclature

C	constant of integration that equals the maximum value of shear stress
Cp	specific heat at constant pressure of fluid
D_H	hydraulic diameter
g	acceleration of gravity
H	distance between plates
KE	kinetic energy
KE/VOL	average kinetic energy per unit volume for flow
k	thermal conductivity of fluid
Pr	Prandtl number
Q	volume flow rate parallel plate duct
Gr	Grashof number
Gr_{eq}	equivalent Grashof' number
Ra	Rayleigh number, $Ra = Gr\,Pr$
Re	Reynolds number
Re_{eq}	equivalent Reynolds number
$T(x)$	temperature distribution
V	mean velocity of upward flow on hot side of convective flow pattern
V_{avg}	average velocity for flow in parallel duct
V_c	upward velocity of hot core
v	local upward velocity in y-direction, a function of x
x	coordinate across gap, $x = 0$ is at hot plate, $x = H$ is at cold plate
β	the thermal coefficient of volumetric expansion of the fluid
$\gamma = \rho g$	weight density
$\Delta p/\Delta L$	pressure gradient along parallel plate duct
ΔT	temperature differential across plates
μ	dynamic viscosity
μE	effective viscosity
ξ	x/H
ξ_1	value of ξ at hotter boundary of hot core
ξ_2	value of ξ at colder boundary of hot core
ρ	mass density of the fluid
τ_{xy}	shear stress in flowing fluid
τ_{MAX}	maximum shear stress in the natural convective fluid flow
$\lvert\tau\rvert_{MAX}$	maximum shear stress in the forced convection fluid flow
τ_{MIN}	minimum shear stress in the flowing fluid
τo	yield point for Bingham material model

transfer of these non-Newtonian fluids. As more becomes know about the relationship between traditional laboratory fluid properties (such as shear rate and yield point) and the consequent convective heat transfer correlations, this will open up opportunities to better design packer fluids to achieve a combination of traditional service roles plus effective insulation. A fundamental analysis of the relationship between fluid properties and heat transfer correlation will enable an understanding of how to design these fluids to fully perform as needed.

These special fluids can have rather large gel strengths which can be detrimental when the fluid has to be pumped out of the well for workover, recompletion, or abandonment. The limits on the gel strength appear to be between the minimum value that provides insulation (usually by preventing convection) and the maximum value that permits pumping and removal of a casing string. When these limits are known, it will be prudent to choose a gel strength at the low end of the range to minimize both cost of the fluid and difficulty removing a string.

The literature on natural convective heat transfer for non-Newtonian fluids is quite limited. Skelland's book [2] gives a good review of the previous research in this area. For the Newtonian fluids we have several excellent sources and reviews of the natural convective flows [3–6]. The presence of secondary and tertiary flows for the flow between parallel plates is reported by Elder [7]. As the temperature difference between the plates increases, the flow patterns go from single cell to multiple cell flow with cells having alternating signs for their circulation. Gill's [8] analytical approach models two boundary layers separated by a core. Some predictions from this analysis are compared with experimental results from Elder. Bejan [9] modified Gill's solution. Bejan's predictions are shown to be in agreement with several published works in the literature [3–6]. Several other very highly cited literature are mainly for the Newtonian fluids [10–15].

In this work the gel is modeled, mathematically, as a Bingham material. A Bingham fluid model has a yield point and a linear shear stress – shear strain rate relation. The salient property is that it takes a finite shear stress to initiate nonrigid-body motion. This finite shear stress is called the gel strength. When a temperature differential exists across parallel plates containing Bingham fluid, the resulting linear temperature variation causes the density to vary. The nonuniform density distribution causes body forces and thus develops stresses. In order for flow to occur, the stresses must be sufficiently large to overcome the gel strength. Consequently, a finite temperature differential is required to initiate convective flow.

Consider an undisturbed Bingham fluid between two, large, vertical, parallel plates that are, initially, at a uniform temperature. There is a hydrostatic pressure that varies in the vertical direction. Owing to the assumption of very large plates (extent of plates $\gg$ gap between plates), the shear stresses vanish everywhere. The fluid flow is assumed to be incompressible. The gap in the plates is sealed around all the edges of the plates and horizontal flow parallel to the plate surfaces is neglected.

The problem considered here is the determination of the stress changes and velocities in the gel when a temperature differential is applied across the plates. Only the steady-state temperature distribution and laminar flow are considered. When nontrivial flow is initiated, there can be no net vertical volume flow at any elevation. Therefore, at each elevation, any sub layer with an upward flow must be accompanied with another sub layer with a downward flow. Velocity components normal to the plates are neglected so that the region in the center of a convection cell is being analyzed. Yang and Yeh [6] present a solution for the mechanical flow of a Bingham material that is used in this paper. In their work the temperature profile is assumed to be linear across the gap and the solution is given in the form of four simultaneous and nonlinear equations.

The hydrostatic component of the stress has no influence on the initiation of nontrivial flow. In the problem considered here, nontrivial flow is initiated by the shear stress that acts vertically on the plate/fluid interface. When the magnitude of this shear stress exceeds the gel strength, nontrivial flow will occur. That is, there is no slip at the plate/fluid interface. The magnitude of the corresponding shear rate equals the excess of the magnitude of the shear stress over the gel strength divided by the plastic viscosity.

When nontrivial flow occurs in this natural convection problem, there are five distinct regions of flow in the gap between the parallel plates. They may be described as follows progressing from the hotter plate to the cooler plate,

- upward velocity, shearing flow
- upward velocity, rigid-body motion (the hot core)
- first upward then downward velocity, shearing flow
- downward velocity, rigid-body motion (the cold core)
- downward velocity, shearing flow

The geometric symmetry of this problem makes possible a simplification of the analysis. Only a section of the gel between one plate and the mid-plane of the gel need be considered in the mathematical analysis.

2. Viscous flow between vertical parallel plates

Consider the steady two-dimensional natural convection of a viscous fluid in the slot shown in Fig. 1. For this vertical enclosure for large aspect ratios $r^* = L/H$, the motion may be assumed to be parallel away from the ends, i.e., $u = 0$ and $v = v(x)$ where u and v are the velocity components in the x- and y-directions. Hence the governing equations with the use of the Boussinesq approximation reduces to

$$-\frac{\mathrm{d}p'}{\mathrm{d}y} + \rho g\beta T' + \frac{\mathrm{d}\tau_{xy}}{\mathrm{d}x} = 0 \tag{1a}$$

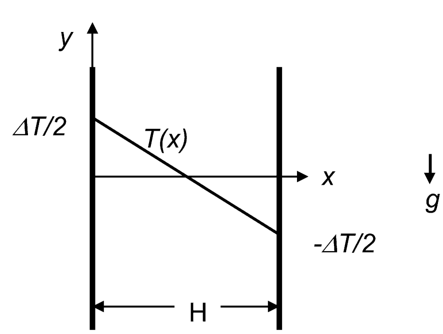


Fig. 1. Schematic of the enclosure.

$$v\frac{\partial T'}{\partial y} = \alpha\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}\right) \tag{1b}$$

where the fluid properties β and α are assumed to be constant, and p' and T' are the deviations of pressure and temperature from their values at the hydrostatic condition at p_0 and T_0, i.e.,

$$p = p_0 + p' \tag{2a}$$

$$T = T_0 + T' \tag{2b}$$

The boundary conditions are

$$\text{at } x = 0, \quad v = 0, \quad T = T_0 + \Delta T/2 \tag{3a}$$

$$\text{at } x = H, \quad v = 0, \quad T = T_0 - \Delta T/2 \tag{3b}$$

For very large aspect ratios $r^* = L/H$, the vertical velocity and the temperature distribution take the asymptotic form at all points except either end of the enclosure. Eqs. (1a) and (1b) reduce to

$$\rho g\beta T' + \frac{\mathrm{d}\tau_{xy}}{\mathrm{d}x} = 0 \tag{4a}$$

$$\frac{\mathrm{d}^2 T'}{\mathrm{d}x^2} = 0 \tag{4b}$$

Eq. (4b) and its boundary conditions give the following linear temperature distribution,

$$T(x) = \Delta T\left(\frac{1}{2} - \frac{x}{H}\right) \tag{5}$$

where the hot plate is at $x = 0$ and the cold plate is at $x = H$. Integrating Eq. (1a), the shear stress is determined by,

$$\tau_{xy} = -\frac{1}{2}\rho g\beta H\Delta T\left(\frac{x}{H} - \left(\frac{x}{H}\right)^2\right) + C \tag{6}$$

2.1. Linear (Newtonian) viscous flow

In order to provide a reference case, we present the solution for the flow in the linear viscous case (gel strength = 0). Consequently, this section is not concerned with a Bingham material. The shear stress distribution given in Eq. (6) is now substituted into the linear viscous flow equation as

$$\mu\frac{\mathrm{d}v}{\mathrm{d}x} = \tau_{xy} = -\frac{1}{2}\rho g\beta H\Delta T\left(\frac{x}{H} - \left(\frac{x}{H}\right)^2\right) + C \tag{7}$$

where v is the vertical velocity in the y direction. This equation may be integrated to obtain,

$$\mu v = -\frac{1}{2}\rho g\beta H\Delta T\left(\frac{1}{2}\left(\frac{x}{H}\right)^2 - \frac{1}{3}\left(\frac{x}{H}\right)^3\right) + Cx + D_1 \tag{8}$$

The boundary conditions given in Eqs. (3c) and (3d) that the velocity vanish at $x = 0$ and $x = H$ yield

$$C = \frac{1}{2}\rho g\beta H\Delta T \tag{9a}$$

$$D_1 = 0 \tag{9b}$$

Then, for the linear viscous flow we have

$$\tau_{xy} = \frac{1}{2}\rho g\beta H\Delta T\left(\left(\frac{x}{H}\right)^2 - \frac{x}{H} + \frac{1}{6}\right) \tag{10}$$

$$\mu v = \frac{1}{2}\rho g\beta H\Delta T\left(\frac{1}{3}\left(\frac{x}{H}\right)^3 - \frac{1}{2}\left(\frac{x}{H}\right)^2 + \frac{1}{6}\left(\frac{x}{H}\right)\right) \tag{11}$$

The mean velocity, V, of the flow is calculated as

$$V = \frac{2}{H}\int_{x=0}^{x=H/2} V\,\mathrm{d}x = \frac{\rho g\beta\Delta T H^2}{192\mu} \tag{12}$$

The maximum shear stress, τ_{MAX} and the minimum shear stress, τ_{MIN} from Eq. (7) are

$$\tau_{\mathrm{MAX}} = C = \frac{1}{12}\rho g\beta H\Delta T \tag{13a}$$

$$\tau_{\mathrm{MIN}} = C - \frac{1}{8}\rho g\beta H\Delta T = -\frac{1}{24}\rho g\beta H\Delta T \tag{13b}$$

The average kinetic energy per unit volume, KE/VOL, is calculated to be

$$KE/VOL = \frac{1}{H}\int_{x=0}^{x=H}\frac{1}{2}\rho v^2\,\mathrm{d}x = \frac{\rho(\rho g\beta H\Delta T)^2 H^2}{15\,120\mu^2} \tag{14}$$

Then we have

$$\frac{KE/VOL}{\tau_{\mathrm{MAX}}} = \frac{1}{1260}Gr \tag{15a}$$

where Grashof number Gr, is defined as

$$Gr = \frac{\rho^2 g\beta\Delta T H^3}{\mu^2} \tag{15b}$$

The next section considers the problem treated in this section when the fluid is modeled as a Bingham material.

2.2. Bingham fluid flow

The constant of integration, C in Eq. (7), is the value of the shear stress at $x = 0$ and is also the maximum shear stress,

$$\tau_{\mathrm{MAX}} = C \tag{16}$$

Fig. 2 shows how τ_{xy} varies across the gap. In order to have nontrivial flow, the absolute value of τ_{xy} must exceed the gel strength at each plate and in a central region. Only the left half of the gel layer is shown in this figure owing to the symmetry of this problem. In the graph, there is a core (the hot core) that behaves as a rigid-body. The boundaries of this core are $x/H = \xi_1$ and $x/H = \xi_2$ as indicated in Fig. 2. The corresponding graph for the right side of the gel layer is a mirror image about $x/H = 0.5$. Let the vertical velocity, parallel to the plates, be $v = v(x)$ so that the constitutive equations in the flowing regions are as follows:

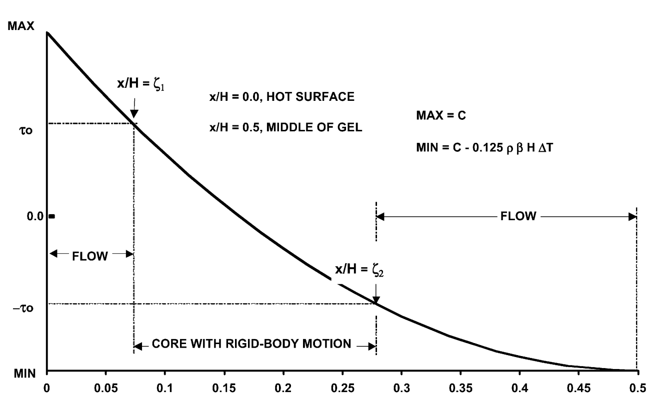


Fig. 2. Shear stress versus position in gel.

$$\text{For } 0 \leqslant \frac{x}{H} \leqslant \xi_1, \quad \mu\frac{\mathrm{d}v}{\mathrm{d}x} = \tau_{xy} - \tau o \tag{17a}$$

$$\text{For } \xi_2 \leqslant \frac{x}{H} \leqslant \frac{1}{2}, \quad \mu\frac{\mathrm{d}v}{\mathrm{d}x} = \tau_{xy} + \tau o \tag{17b}$$

Consider first the region, $0 \leqslant x/H \leqslant \xi_1$. In this region

$$\text{at } \frac{x}{H} = \xi_1, \quad \tau_{xy} = \tau o \quad \text{and}$$

$$\xi_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8\left(\frac{C}{\tau o} - 1\right)\frac{\tau o}{\rho g H}\frac{1}{\beta\Delta T}} \tag{18}$$

Using the constitutive equation we have

$$\mu\frac{\mathrm{d}v}{\mathrm{d}x} = \tau_{xy} - \tau o = -\frac{1}{2}\rho g\beta H\Delta T\left(\frac{x}{H} - \left(\frac{x}{H}\right)^2\right) + C - \tau o \tag{19}$$

and integrating Eq. (19) and using the boundary condition $v = 0$ at $x/H = 0$ results

$$\frac{\mu v}{\tau o H} = -\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{1}{2}\left(\frac{x}{H}\right)^2 - \frac{1}{3}\left(\frac{x}{H}\right)^3\right) + \left(\frac{C}{\tau o} - 1\right)\frac{x}{H} \tag{20}$$

Since $v \equiv V_{\mathrm{c}}$ at $x/H = \xi_1$, we have

$$\frac{\mu V_{\mathrm{c}}}{\tau o H} = -\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{1}{2}\xi_1^2 - \frac{1}{3}\xi_1^3\right) + \left(\frac{C}{\tau o} - 1\right)\xi_1 \tag{21}$$

Now consider the region, $\xi_2 \leqslant x/H \leqslant 1/2$. In this region

$$\text{at } \frac{x}{H} = \xi_2, \quad \tau_{xy} = -\tau o \quad \text{and}$$

$$\xi_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8\left(\frac{C}{\tau o} + 1\right)\frac{\tau o}{\rho g H}\frac{1}{\beta\Delta T}} \tag{22}$$

Using the constitutive equation gives

$$\mu\frac{\mathrm{d}v}{\mathrm{d}x} = \tau_{xy} + \tau o = -\frac{1}{2}\rho g\beta H\Delta T\left(\frac{x}{H} - \left(\frac{x}{H}\right)^2\right) + C + \tau o \tag{23}$$

and integrating Eq. (23) and using the boundary condition $v = 0$ when $x/H = 1/2$ results

$$\frac{\mu v}{\tau o H} = -\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{1}{2}\left(\frac{x}{H}\right)^2 - \frac{1}{3}\left(\frac{x}{H}\right)^3 - \frac{1}{12}\right) + \left(\frac{C}{\tau o} + 1\right)\left(\frac{x}{H} - \frac{1}{2}\right) \tag{24}$$

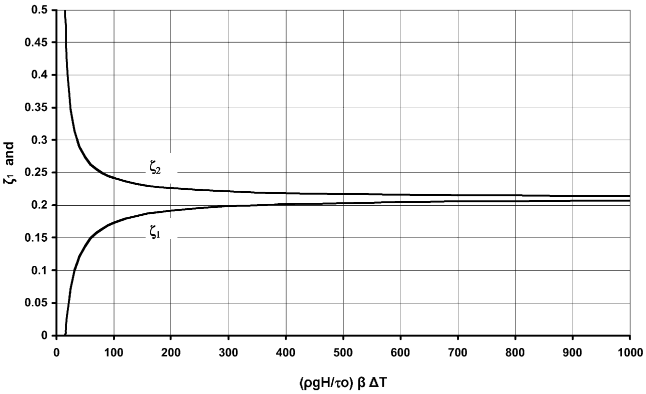


Fig. 3. ζ_1 and ζ_2 versus $(\rho g H/\tau o)\beta\Delta T$.

Since $v \equiv V_c$ at $x/H = \xi_2$, we have

$$\frac{\mu V_c}{\tau o H} = -\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{1}{2}\xi_2^2 - \frac{1}{3}\xi_2^3 - \frac{1}{12}\right) + \left(\frac{C}{\tau o}+1\right)\left(\xi_2 - \frac{1}{2}\right) \tag{25}$$

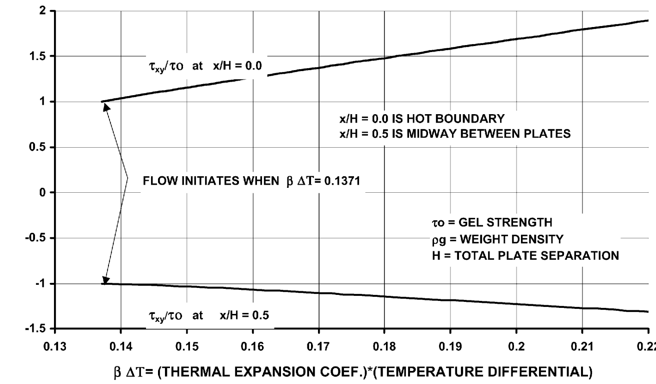
The average velocity V, in the region $0 < x/H < 0.5$ is found by integration, in dimensionless form,

$$\frac{\mu V}{\tau o H} = -\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{\xi_1^2}{2} - \frac{\xi_1^3}{6} - \frac{\xi_1^4}{12} - \frac{5}{192} + \frac{\xi_2}{12} - \frac{\xi_2^3}{6} + \frac{\xi_2^4}{12}\right) + 2\frac{C}{\tau o}\left(\frac{\xi_1^2}{2} + \xi_1(\xi_2-\xi_1) - \frac{\xi_2^2}{2} + \frac{\xi_2}{2} - \frac{1}{8}\right) - 2\left(\frac{\xi_1^2}{2} + \xi_1(\xi_2-\xi_1) + \frac{\xi_2^2}{2} - \frac{\xi_2}{2} + \frac{1}{8}\right) \tag{26}$$

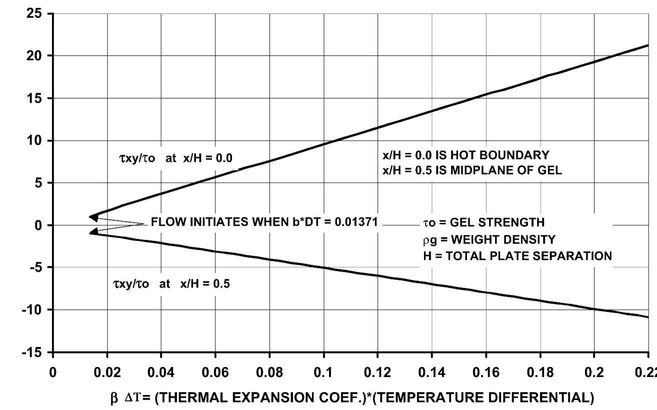
In both regions considered above, the velocity at the core, moving as a rigid body, is designated as V_c. Since velocity must be continuous across the gel layer, the values of V_c must be the same for both regions as well as for the core. The condition that V_c be the same in both regions leads to

$$-\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{1}{2}\xi_1^2 - \frac{1}{3}\xi_1^3\right) + \left(\frac{C}{\tau o}-1\right)\xi_1 = -\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T\left(\frac{1}{2}\xi_2^2 - \frac{1}{3}\xi_2^3 - \frac{1}{12}\right) + \left(\frac{C}{\tau o}+1\right)\left(\xi_2 - \frac{1}{2}\right) \tag{27}$$

This is the equation that determines the dimensionless constant of integration, $C/\tau o$. As ξ_1 and ξ_2 are functions of C, this equation must be solved numerically, once $(\rho g H/\tau o)\beta\Delta T$ is specified. Values for $(\rho g H/\tau o)\beta\Delta T$ in typical casing annuli are in the neighborhood of 5 to 100. Fig. 3 gives the dependence of ξ_1 and ξ_2 on $(\rho g H/\tau o)\beta\Delta T$. As $(\rho g H/\tau o)\beta\Delta T$ increases, ξ_1 and ξ_2 approach one another and the volume of the core, moving as a rigid body, becomes smaller.



(a)



(b)

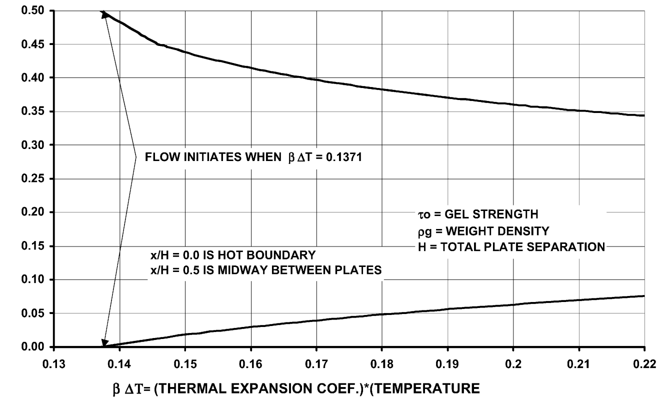
Fig. 4. (a) Maximum and minimum dimensionless shear stress versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.008571$. (b) Maximum and minimum dimensionless shear stress versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.0008571$.

2.3. Illustrative problem

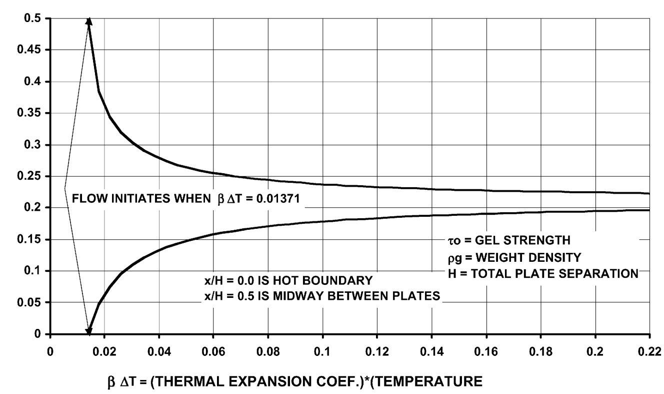
The following parameters are chosen for an illustrative problem;

$\gamma = 70$ pounds per cubic foot $= 9.358$ pounds per gallon
$H = 1$ inch
$\tau o = 5$ pounds per 100 square feet
$\beta = 0.0005$ reciprocal degrees Fahrenheit
$\mu = 3.0$E-7 psi-seconds

these parameters result in $\tau o/\gamma H = 0.008571$. The value of ΔT is varied from 0 to 440 degrees Fahrenheit so that $\beta\Delta T$ varies from 0 to 0.22. Fig. 4(a) indicates how the shear stresses at $x/H = 0$ and at $x/H = 0.5$ are altered when the value of $\beta\Delta T$ changes. There is no flow until $\beta\Delta T = 0.1371$ ($\Delta T = 274.2$ °F). At this initiation point, the shear stresses are τo at $x/H = 0$ and $-\tau o$ at $x/H = 0.5$. Fig. 5(a) gives the locations of the boundaries of the core undergoing rigid body motion. As noted earlier, the thickness of the core reduces as $\beta\Delta T$ increases in the flowing region of the curves. Fig. 6(a) shows the dimensionless core velocity as a function of $\beta\Delta T$. Note that a dimensionless core velocity of 0.033 corresponds to a core velocity of 3.183 feet per second. Fig. 7(a) gives the percentage of the gel layer that is undergoing rigid-body motion as a function

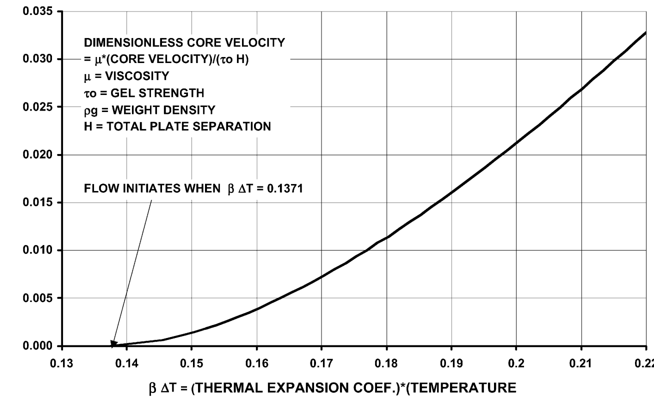


(a)

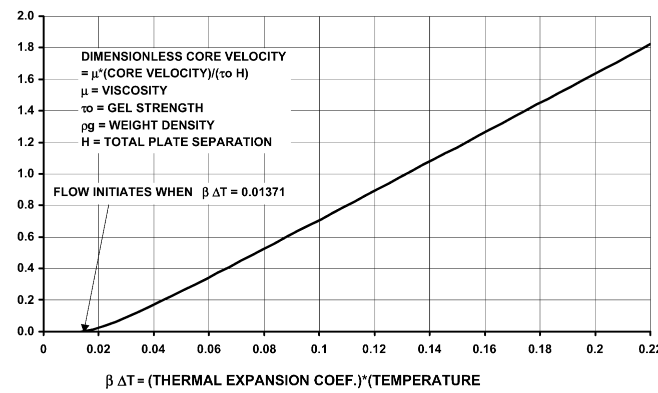


(b)

Fig. 5. (a) Position of core boundaries versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.008571$. (b) Position of core boundaries versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.0008571$.



(a)



(b)

Fig. 6. (a) Dimensionless core velocity versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.008571$. (b) Dimensionless core velocity versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.0008571$.

of $\beta\Delta T$. For this illustrative problem, the percentage is seen to be about 53% when $\beta\Delta T = 0.22$ ($\Delta T = 440\ ^\circ\text{F}$).

This illustrative problem is typical for many wells at installation. Depending upon the gel, the gel strength may decrease with time and thus give less resistance to convective heat transfer. To illustrate this effect, the above illustrative problem was repeated with the same parameters except that τo was reduced by a factor of ten. That is, τo is equal to 0.5 pounds per 100 square feet. Figs. 4(b), 5(b), 6(b) and 7(b) correspond to Figs. 4(a), 5(a), 6(a) and 7(a), respectively, for the initial illustrative problem and $\gamma = \rho g$. It is interesting to note the rather dramatic changes in the solution when the gel strength is reduced.

2.4. Limits on the validity of the laminar flow solution for Bingham fluid

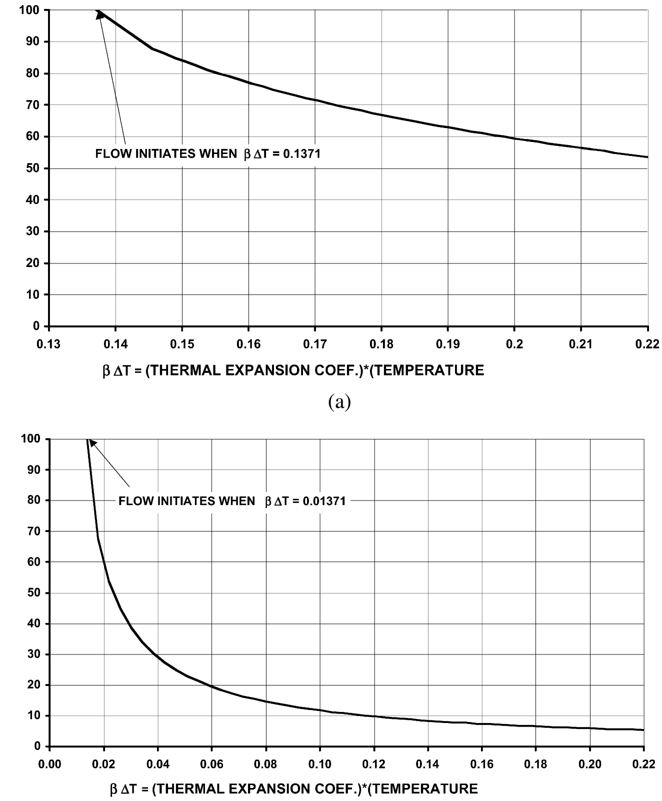
In the case of linear viscous fluids, the transition from laminar flow to turbulent flow is governed by the value of the dimensionless Reynolds number, *Re*. This transition in a duct flow has been well established experimentally and occurs for $Re < 2000$ [4]. The physical basis for choosing the Reynolds number as a turbulent flow indicator is that it is a measure of the ratio in laminar flow of the average kinetic energy per unit volume of the flow to the maximum shear stress occurring in the flow. The underlying notion is that, when this ratio becomes large enough, any disruption in the flow (however small) will cause the kinetic energy density to "overwhelm" the viscous shear stress and lead to turbulent flow. The flow remains laminar in a thin laminar sublayer near the wall but the primary characteristics of the flow, such as axial pressure gradient, are controlled mainly by momentum influences rather than viscous influences. It is the purpose of this section to use this fundamental understanding to derive a criterion for the transition from laminar to turbulent flow when the flowing fluid is a Bingham material.

To begin, the expression for the Reynolds number for linear viscous flow in a duct (or between parallel plates) is shown to be proportional to the ratio of average kinetic energy per unit volume to the maximum viscous shear stress in conventional laminar duct flow. The definition of the Reynolds number for duct flow is

$$Re \equiv \frac{\rho V_{\text{avg}} D_{\text{H}}}{\mu} \tag{28}$$

where ρ is the mass density of the fluid, V_{avg} the volume rate of flow per unit duct cross-sectional area, D_{H} is the hydraulic diameter of the duct and μ the dynamic viscosity of fluid.

The linear viscous flow solution for laminar flow between parallel plates separated by $H/2$ (assuming an imaginary wall at the middle of the gap H) without the gravitational forces and for a constant pressure gradient, Δp in the flow direction over a channel height ΔL, using the same coordinates illustrated in Fig. 1 is as follows:



(a)

(b)

Fig. 7. (a) Percent (%) core gap versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.008571$. (b) Percent (%) core gap versus $\beta\Delta T$ for $\tau o/(\rho g H) = 0.0008571$.

The velocity distribution:

$$v(x) = \frac{x(H/2 - x)}{2\mu}\frac{\Delta p}{\Delta L} \tag{29a}$$

The flow rate per unit depth:

$$Q = \frac{(H/2)^3}{12\mu}\frac{\Delta p}{\Delta L} \tag{29b}$$

The average velocity:

$$V_{\mathrm{avg}} = \frac{Q}{H/2} = \frac{(H/2)^2}{12\mu}\frac{\Delta p}{\Delta L} \tag{29c}$$

The velocity distribution in terms of the average velocity:

$$v(x) = \frac{6V_{\mathrm{avg}}}{(H/2)^2}x\left(\frac{H}{2} - x\right) \tag{29d}$$

The shear stress distribution:

$$\tau = -\mu\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{6\mu V_{\mathrm{avg}}}{(H/2)^2}\left(\frac{H}{2} - 2x\right) \tag{29e}$$

The maximum shear stress:

$$|\tau|_{\mathrm{MAX}} = \left|\tau(x=0)\right|_{\mathrm{MAX}} = -\frac{6\mu V_{\mathrm{avg}}}{H/2} = -\frac{12\mu V_{\mathrm{avg}}}{H} \tag{29f}$$

where $\Delta p/\Delta L$ is the pressure gradient along the duct length L, Q is the volume rate of flow between the parallel plates, V_{avg} is the average velocity of flowing fluid, v the velocity at distance x, and τ is the viscous shear stress at distance x.

The kinetic energy per unit depth becomes

$$KE = \int_{x=0}^{x=H/2} \frac{1}{2}\rho v(x)^2\,\mathrm{d}x = \frac{3}{5}\rho V_{\mathrm{avg}}^2\frac{H}{2} \tag{30}$$

and the average kinetic energy per unit volume for a unit height, KE/VOL is

$$KE/VOL = \frac{KE}{H/2} = \frac{3}{5}\rho V_{\mathrm{avg}}^2 \tag{31}$$

Since the maximum value of the viscous shear stress, τ_{MAX}, is

$$|\tau|_{\mathrm{MAX}} = \frac{6\mu V_{\mathrm{avg}}}{H/2} = \frac{12\mu V_{\mathrm{avg}}}{H} \tag{32}$$

Then we have

$$\frac{KE/VOL}{|\tau|_{\mathrm{MAX}}} = \frac{1}{20}\frac{\rho V_{\mathrm{avg}} H}{\mu} = \frac{1}{20}Re \tag{33}$$

where the hydraulic diameter $D_{\mathrm{H}} = H$ for half the distance between the plates $H/2$. Here it is shown that the Reynolds number is proportional to the ratio of average kinetic energy per unit volume to the maximum viscous shear stress in laminar flow.

Now consider the case of a Bingham fluid flowing between parallel plates. If the physical reasoning that is applied above is used for the Bingham fluid case, then a criterion may be developed for the transition from laminar flow to turbulent flow. The solution obtained above will now be used to develop the criterion. That is, an "equivalent Reynolds number", Re_{eq}, is derived that is defined in terms of the average kinetic energy per unit volume and the maximum shear stress as

$$Re_{\mathrm{eq}} = 20\left(\frac{KE/VOL}{|\tau|_{\mathrm{MAX}}}\right)_{\mathrm{Bingham}} \tag{34}$$

The above analysis showed that the parameter, C, is the maximum value of the shear stress so it is only necessary to find the average kinetic energy per unit volume, KE/VOL. The derivation is sketched below.

$$\begin{aligned} KE/VOL &= \frac{1}{H}\frac{\rho\tau o^2 H^2}{\mu^2}\int_{x=0}^{x=H/2}\left(\frac{\mu v}{\tau o H}\right)^2\mathrm{d}x \\ &= \frac{\rho\tau o^2 H^2}{\mu^2}\Bigg[\int_{x/H=0}^{x/H=\xi_1}\left(\frac{\mu v}{\tau o H}\right)^2\mathrm{d}\left(\frac{x}{H}\right) \\ &\quad + \left(\frac{\mu V}{\tau o H}\right)^2(\xi_2 - \xi_1) \\ &\quad + \int_{x/H=\xi_2}^{x/H=1/2}\left(\frac{\mu v}{\tau o H}\right)^2\mathrm{d}\left(\frac{x}{H}\right)\Bigg] \end{aligned} \tag{35}$$

This integral is separated into three pieces so that,

$$(KE/VOL)_{\mathrm{Bingham}} = KE/VOL1 + KE/VOL2 + KE/VOL3 \tag{36}$$

where

$$KE/VOL1 = \frac{\rho\tau o^2 H^2}{\mu^2}\left[A^2\left(\frac{1}{20}\xi_1^5 - \frac{1}{18}\xi_1^6 + \frac{1}{63}\xi_1^7\right) + 2AB_1\left(\frac{1}{8}\xi_1^4 - \frac{1}{15}\xi_1^5\right) + B_1^2\frac{1}{3}\xi_1^3\right]$$

and

$$A = -\frac{1}{2}\frac{\rho g H}{\tau o}\beta\Delta T \quad \text{and} \quad B_1 = \frac{C}{\tau o} - 1$$

$$KE/VOL2 = \frac{\rho\tau o^2 H^2}{\mu^2}\left[A^2\left(\frac{1}{2}\xi_1^2 - \frac{1}{3}\xi_1^3\right)^2 + 2AB_1\left(\frac{1}{2}\xi_1^3 - \frac{1}{3}\xi_1^4\right) + B_1^2\xi_1^2\right](\xi_2 - \xi_1)$$

$$\begin{aligned}KE/VOL3 = \frac{\rho\tau o^2 H^2}{\mu^2}\Bigg\{ & A^2\left[\frac{1}{20}\left(\frac{1}{32} - \xi_2^2\right) + \frac{1}{63}\left(\frac{1}{128} - \xi_2^7\right) + \frac{1}{144}\left(\frac{1}{2} - \xi_2\right) - \frac{1}{18}\left(\frac{1}{64} - \xi_2^6\right) - \frac{1}{36}\left(\frac{1}{8} - \xi_2^3\right) + \frac{1}{72}\left(\frac{1}{16} - \xi_2^4\right)\right] \\ & + 2AB_2\left[\frac{1}{8}\left(\frac{1}{16} - \xi_2^4\right) - \frac{1}{15}\left(\frac{1}{32} - \xi_2^5\right) - \frac{1}{24}\left(\frac{1}{4} - \xi_2^2\right) - \frac{1}{12}\left(\frac{1}{8} - \xi_2^3\right) + \frac{1}{24}\left(\frac{1}{16} - \xi_2^4\right) + \left(\frac{1}{2} - \xi_2\right)\right] \\ & + B_2^2\left[\frac{1}{3}\left(\frac{1}{8} - \xi_2^3\right) - \frac{1}{2}\left(\frac{1}{4} - \xi_2^2\right) + \frac{1}{4}\left(\frac{1}{2} - \xi_2\right)\right]\Bigg\}\end{aligned}$$

and $B_2 = C/\tau o + 1$.

We also use Eq. (15a) to define an equivalent Grashof number, Gr_{eq} as

$$Gr_{eq} = 1260\left(\frac{KE/VOL}{\tau_{MAX}}\right)_{Bingham} \tag{37}$$

It should be noted that for the problem we have under consideration if we assume $\tau_{MAX} \sim |\tau|_{MAX} = C$ for Eq. (34). Then from Eqs. (34) and (37), we have

$$Gr_{eq} \sim 63\,Re_{eq} \tag{38}$$

An effort is made in the following part of this report to relate this work to the Batchelor's paper [15]. The main emphasis in the Batchelor paper is on air cavities in windows and between double walls in buildings. The work is, of course, for Newtonian viscous fluids. In the analysis, Batchelor uses two criteria to determine if the laminar-turbulent transition for the convecting flow has occurred. In order to have natural convective turbulent flow between infinitely long parallel plates both criteria must be met. The criteria are,

1. The value of the Rayleigh number Ra, based on vertical length, L, must exceed 10^9. That is,

$$Ra = Gr\,Pr > \left(\frac{H}{L}\right)^3 10^9 \tag{39}$$

2. The Reynolds number must exceed 300.

In this work, the Grashof number is 63 times the Reynolds' number so this condition, in terms of the Rayleigh number with $Pr = 0.7326$ is,

$$Ra = Gr\,Pr = 63\,Re\,Pr = 18{,}900\,Pr = 13{,}846$$

In Batchelor's work the 13,846 in the above equation is 13,700 as taken from Fig. 4 of Batchelor's paper [15]. That figure shows that when the cube root of the Rayleigh number is specified and it satisfies the second turbulent flow condition, a value of L/H is determined from the first turbulent flow condition. This is the height-to-width ratio for the laminar-turbulent transition. When the actual ratio is less than this value, the flow is laminar. When the actual ratio is greater than this, the flow is turbulent. When the second turbulent flow condition is not satisfied, the flow will be laminar for any height-to-width ratio. It is interesting to note that the curves for the two conditions intersect at a height-to width-ratio of 42.

3. Conclusions

The most important conclusion is that the influence of gel strength on convective heat transfer rate in oil industry applications is quite strong. This paper would help to the development free convective heat transfer correlation equation and the pressure drop calculations for the non-Newtonian packer fluid. It will link fluid thermal performance to the laboratory properties and will enable the creation of "designer fluids". In addition being able to reduce the heat transfer is critical to the longevity and economic viability of wells. Another conclusion is that the temperature differential required to initiate flow can be appreciable in practical applications. These conclusions are sufficient to justify the development of a new design procedure for gels. The mathematical model for the fluid used here is the Bingham material. Attitudes differ concerning the appropriateness of this model. The model was developed because it is simple enough to permit explicit mathematical solutions for flow problems. There are other characteristics of real gels that are not captured by this model, as given by Darley and Gray [16]. The most significant shortcoming of the Bingham material is for transient problems. When the gel strength is derived from electrical forces between particles, the gel strength is a function of the history of the shearing rate in the material. In some formulations this phenomenon is accounted for by allowing the gel strength to be a function of the deformation rate history. This breakdown of the gel strength with flow will alter the flow solution. If fluids are considered that have considerable gel strength breakdown, the results presented here may need to be extended to account for this behavior. Another shortcoming of the Bingham material model is the way the gel strength is taken as the minimum stress for which there can be deformation. It is true that real gels exhibit behavior that is like a very high viscosity viscous fluid when the stress is below the gel strength. Other models for gels have properties that are tied to the chemical structure of the fluid. These models are very helpful during the development and modification of gels. The investigation presented here is concerned only with mechanical properties and needs to be as mathematically elementary as can be justified.

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